OCR
Oxford Cambridge and RSA

Monday 05 October 2020

A Level Further Mathematics B (MEI)

Y420/01 Core Pure

Worked Solutions

Printed Answer Booklet

Time allowed: 2 hours 40 minutes

You must have:
- Question Paper Y420/01 (inside this document)
- the Formulae Booklet for Further Mathematics B (MEI)
- a scientific or graphical calculator

R red level
- longer questions (6+ marks)
- higher level problem solving
- harder A Level content

A amber level
- shorter questions (3-6 marks)
- low level problem solving
- harder AS/easier A Level content.

G green level
- short questions (1-3 marks)
- minimal problem solving
- AS/easier A Level content.

E explanation

Section A (36 marks)
Q1: Series (as)
Q2: Matrices (as)
Q3: Calculus (a level)
Q4: Algebra (as)
Q5: Polar Coordinates (a level)
Q6: Complex Numbers (as)

Section B (108 marks)
Q7: Proof (as)
Q8: Vectors (a level)
Q9: Matrices (as)
Q10: Calculus (a level)
Q11: Complex Numbers (as)
Q12: Complex Numbers (a level)
Q13: Hyperbolic functions (a level)
Q14: Differential Equations (a level)
Q15: Matrices (a level), Vectors (a level)
Q16: Differential Equations (a level)

Grade Boundaries

Grade	A*	A	B	C	D	E	U
Mark / 144	119	93	76	60	44	28	0
Scaled/ 180	149	116	95	75	55	35	0

x 1.25

note: the scaled score is added to the scores in the other modules to find an overall grade, not the raw mark

Turn over

Section A (36 marks)

A

1 Using standard summation of series formulae, determine the sum of the first n terms of the series

$(1\times2\times4)+(2\times3\times5)+(3\times4\times6)+\ldots,$

where n is a positive integer. Give your answer in fully factorised form. **[6]**

First we need to write this series in a general form.

$$\sum_{r=1}^{n}\left(r(r+1)(r+3)\right)$$

expand out brackets

$$=\sum_{r=1}^{n}(r^3+4r^2+3r)$$

using $\sum ar^3+br^2+cr = a\sum r^3+b\sum r+c\sum r$

$$=\sum_{r=1}^{n}r^3+4\sum_{r=1}^{n}r^2+3\sum_{r=1}^{n}r$$

Now, in the formula book, $\sum_{r=1}^{n}r^3=\frac{1}{4}n^2(n+1)^2$ and $\sum_{r=1}^{n}r^2=\frac{1}{6}n(n+1)(2n+1)$.

Also recall $\sum_{r=1}^{n}r=\frac{1}{2}n(n+1)$

$$=\frac{1}{4}n^2(n+1)^2+4\left(\frac{1}{6}n(n+1)(2n+1)\right)+3\left(\frac{1}{2}n(n+1)\right)$$
$$=\frac{1}{12}n(n+1)\left[3n(n+1)+8(2n+1)+18\right]$$
$$=\frac{1}{12}n(n+1)(3n^2+3n+16n+8+18)$$
$$=\frac{1}{12}n(n+1)(3n^2+19n+26)$$
$$=\frac{1}{12}n(n+1)(n+2)(3n+13)$$

always check if quadratics will factorise. You will lose marks if your answer is not in it's simplest form.

G

2 (a) The matrices $\mathbf{M} = \begin{pmatrix} 0 & 1 & a \\ 1 & b & 0 \end{pmatrix}$ and $\mathbf{N} = \begin{pmatrix} b & -5 \\ -1 & c \\ -1 & 1 \end{pmatrix}$ are such that $\mathbf{MN} = \mathbf{I}$.

Find a, b and c. **[5]**

First we need to find MN.

$$MN = \begin{pmatrix} 0 & 1 & a \\ 1 & b & 0 \end{pmatrix}\begin{pmatrix} b & -5 \\ -1 & c \\ -1 & 1 \end{pmatrix}$$

M is 2×3 and N is 3×2 so the product MN will be a 2×2 matrix

$$= \begin{pmatrix} 0(b) - 1(1) + a(-1) & 0(-5) + 1(c) + a(1) \\ 1(b) + b(-1) + 0(-1) & 1(-5) + b(c) + 0(1) \end{pmatrix}$$

$$= \begin{pmatrix} -1-a & c+a \\ 0 & bc-5 \end{pmatrix}$$

We know $MN = I$. So $\begin{pmatrix} -1-a & c+a \\ 0 & bc-5 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$-1-a=1 \Rightarrow a=-2$

$c+a=0 \Rightarrow c-2=0 \Rightarrow c=2$

$-5+bc=1 \Rightarrow -5+2b=1 \Rightarrow b=3$

hence $a=-2$, $b=3$, $c=2$

E

(b) State with a reason whether or not **N** is the inverse of **M**. **[1]**

M is not a square matrix and hence has no inverse.

G1

3 In this question you must show detailed reasoning.

Find $\int_0^{\frac{1}{3}} \frac{1}{\sqrt{4-9x^2}}\,dx$, expressing your answer in terms of π. **[4]**

We can rewrite this integral into a standard form in the formula book.

$$\int_0^{\frac{1}{3}} \frac{1}{\sqrt{4-9x^2}}\,dx = \frac{1}{\sqrt{9}}\int_0^{\frac{1}{3}} \frac{1}{\sqrt{4/9 - x^2}}\,dx$$

From the formula book we know $\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \arcsin\left(\frac{x}{a}\right) + c$

Hence $= \frac{1}{3}\int_0^{\frac{1}{3}} \frac{1}{\sqrt{\left(\frac{2}{3}\right)^2 - x^2}}\,dx$

using standard result

$= \frac{1}{3}\left[\arcsin\left(\frac{x}{2/3}\right)\right]_0^{\frac{1}{3}}$

$= \frac{1}{3}\left[\arcsin\left(\frac{3x}{2}\right)\right]_0^{\frac{1}{3}}$

subbing in limits

$= \frac{1}{3}\left(\arcsin\left(\frac{3}{2}\times\frac{1}{3}\right) - \arcsin 0\right)$

$= \frac{1}{3}\left(\arcsin\frac{1}{2} - 0\right)$

$= \frac{1}{3}\left(\frac{\pi}{6}\right) = \frac{\pi}{18}$ //

A

4 The roots of the equation $2x^3 - 5x + 7 = 0$ are α, β and γ.

(a) Find $\frac{1}{\alpha}+\frac{1}{\beta}+\frac{1}{\gamma}$. **[4]**

Let's rewrite $\frac{1}{\alpha}+\frac{1}{\beta}+\frac{1}{\gamma}$

$$\frac{1}{\alpha}+\frac{1}{\beta}+\frac{1}{\gamma} = \frac{\beta\gamma}{\alpha\beta\gamma} + \frac{\alpha\gamma}{\alpha\beta\gamma} + \frac{\alpha\beta}{\alpha\beta\gamma}$$

$$= \frac{\beta\gamma + \alpha\gamma + \alpha\beta}{\alpha\beta\gamma} = \frac{\sum\alpha\beta}{\sum\alpha\beta\gamma}$$

Now we can use the fact that $\sum\alpha\beta = \frac{c}{a}$ and $\sum\alpha\beta\gamma = -\frac{d}{a}$ to find $\frac{1}{\alpha}+\frac{1}{\beta}+\frac{1}{\gamma}$

$$\frac{1}{\alpha}+\frac{1}{\beta}+\frac{1}{\gamma} = \frac{c/a}{-d/a} = -\frac{c}{d} = -\frac{(-5)}{7} = \frac{5}{7}$$

$$\therefore \frac{1}{\alpha}+\frac{1}{\beta}+\frac{1}{\gamma} = \frac{5}{7}$$

G

(b) Find an equation with integer coefficients whose roots are $2\alpha-1$, $2\beta-1$ and $2\gamma-1$. **[4]**

If we want an equation in terms of y with roots $2\alpha-1$, $2\beta-1$ and $2\gamma-1$, we can write that

$$y = 2x - 1$$

$$\Rightarrow x = \frac{1}{2}(y+1)$$

So we can substitute $x = \frac{1}{2}(y+1)$ into the equation in x.

$$2\left(\frac{1}{2}(y+1)\right)^3 - 5\left(\frac{1}{2}(y+1)\right) + 7 = 0$$

expand via binomial expansion

$$\frac{2}{8}(y+1)^3 - \frac{5}{2}(y+1) + 7 = 0$$

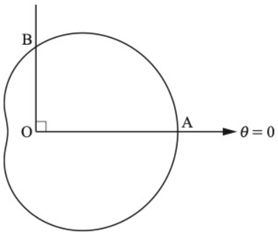
$$\times 4: \quad y^3 + 3y^2 + 3y + 1 - 10(y+1) + 28 = 0$$

$$y^3 + 3y^2 + 3y + 1 - 10y - 10 + 28 = 0$$

$$y^3 + 3y^2 - 7y + 19 = 0$$

5 Fig. 5 shows the curve with polar equation $r = a(3+2\cos\theta)$ for $-\pi \leqslant \theta \leqslant \pi$, where a is a constant.

G



5(a) (a) Write down the polar coordinates of the points A and B. [2]

at A, $\theta = 0 \Rightarrow r = a(3+2\cos 0) = 5a$

so A is $(5a, 0)$

at B, $\theta = \frac{\pi}{2} \Rightarrow r = a(3+2\cos\frac{\pi}{2}) = a(3+2(0)) = 3a$

so B is $(3a, \frac{\pi}{2})$

E

(b) Explain why the curve is symmetrical about the initial line. [2]

$\cos(-\theta) = \cos\theta$

hence the value of r for θ is the same value for $-\theta$.

G

(c) **In this question you must show detailed reasoning.**

Find in terms of a the exact area of the region enclosed by the curve. [4]

In formula book the area $= \frac{1}{2}\int r^2\, d\theta$

so

$$\text{area} = \frac{1}{2}\int_{-\pi}^{\pi} (a(3+2\cos\theta))^2\, d\theta$$

$$= \frac{1}{2}\int_{-\pi}^{\pi} a^2(3+2\cos\theta)^2\, d\theta$$

$$= \frac{a^2}{2}\int_{-\pi}^{\pi} (9 + 12\cos\theta + 4\cos^2\theta)\, d\theta$$

using $\cos^2\theta = \frac{1}{2}(1+\cos 2\theta)$

$$= \frac{a^2}{2}\int_{-\pi}^{\pi} 9 + 12\cos\theta + 4\left(\tfrac{1}{2}(1+\cos 2\theta)\right) d\theta$$

$$= \frac{a^2}{2}\int_{-\pi}^{\pi} 11 + 12\cos\theta + 2\cos 2\theta\, d\theta$$

$\int \cos 4\theta\, d\theta = \frac{1}{4}\sin 4\theta + c$

$$= \frac{a^2}{2}\left[11\theta + 12\sin\theta + \sin 2\theta\right]_{-\pi}^{\pi}$$

$$= \frac{a^2}{2}\left(11\pi + 12\sin\pi + \sin 2\pi - (11(-\pi) + 12\sin(-\pi) + \sin(-2\pi))\right)$$

$$= \frac{a^2}{2}\left(11\pi + 12(0) + 0 - (-11\pi) - 12(0) - (0)\right)$$

$$= \frac{a^2}{2}(22\pi) = 11\pi a^2 \text{ units}^2$$

G

6 The complex number z satisfies the equation $z^2 - 4\mathrm{i}z^* + 11 = 0$.

Given that $\mathrm{Re}(z) > 0$, find z in the form $a + b\mathrm{i}$, where a and b are real numbers. **[4]**

First we define a general complex number z.

let $z = a + bi$. So $z^* = a - bi$

Now we sub this into our equation

$(a+bi)^2 - 4i(a-bi) = -11$

$a^2 + 2abi + b^2(i)^2 - 4ai + 4b(i)^2 = -11$

$a^2 + 2abi + b^2(-1) - 4ai + 4b(-1) = -11$

$a^2 + 2abi - b^2 - 4ai - 4b = -11$

now we need to seperate out the LHS into the real and imaginary part.

$a^2 - b^2 - 4b + (2ab - 4a)i = -11$

The RHS has no imaginary part and a real part of -11

so $2ab - 4a = 0$

$a(2b - 4) = 0$

$\cancel{a = 0}$, $2b = 4$

discard as $\mathrm{Re}(z) > 0 \Rightarrow a > 0$

$b = 2$

Imaginary Part

$a^2 - (2)^2 - 4(2) = -11$

$a^2 - 12 = -11$

$a^2 = 1 \Rightarrow a = \pm 1$

$\mathrm{Re}(z) > 0 \Rightarrow a > 0$. so $a = 1$

Real Part

hence $a = 1$, $b = 2 \Rightarrow z = 1 + 2i$

Section B (108 marks)

A

7 Prove by mathematical induction that $\sum_{r=1}^{n}(r \times r!) = (n+1)! - 1$ for all positive integers n. **[6]**

Step one : base case

When $n=1$, $LHS = \sum_{r=1}^{1}(r \times r!) = 1 \times 1! = 1 \times 1 = 1$

$RHS = (1+1)! - 1 = 2! - 1 = 2 - 1 = 1$

$\therefore$ as $LHS = RHS$, so true for $n=1$.

Step two : assumption

Assume true for $n=k$, so $\sum_{r=1}^{k}(r \times r!) = (k+1)! - 1$

Step three : inductive step

Using the assumed result for $n=k$,

$$\sum_{r=1}^{k+1}(r \times r!) = \sum_{r=1}^{k}(r \times r!) + \left((k+1) \times (k+1)!\right)$$

$$= (k+1)! - 1 + (k+1) \times (k+1)!$$

factorise out $(k+1)!$

$$= (k+1)!\left[1 + (k+1)\right] - 1$$

$(k+1)! = 1 \times 2 \times \ldots \times k \times (k+1)$

$$= (k+1)!(k+2) - 1$$

so $(k+1)!(k+2)$

$$= (k+2)! - 1$$

$= 1 \times 2 \times \ldots \times k \times (k+1) \times (k+2)$

$= (k+2)!$

$$= ((k+1)+1)! - 1 \quad \therefore \text{ true for } n = k+1$$

show clearly that the result follows for $n=k+1$

Step four : conclusion

If the result is true for $n=k$, it is true for $n=k+1$. Since it is true for $n=1$, it is true for all positive integer values of n.

direction vectors

A

8 (a) Given that the lines $\mathbf{r} = \begin{pmatrix} 0 \\ 2 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ 1 \\ 3 \end{pmatrix}$ and $\mathbf{r} = \begin{pmatrix} -1 \\ 2 \\ k \end{pmatrix} + \mu \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix}$ meet, determine k. **[5]**

First we need to set each line equation equal to each other and form a set of equations.

$$\begin{pmatrix} 0 \\ 2 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ 1 \\ 3 \end{pmatrix} = \begin{pmatrix} -1 \\ 2 \\ k \end{pmatrix} + \mu \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix}$$

(A) $0 - \lambda = -1 + 2\mu \Rightarrow \lambda + 2\mu = 1$

(B) $2 + \lambda = 2 + 3\mu \Rightarrow \lambda - 3\mu = 0$

(C) $2 + 3\lambda = k + 4\mu \Rightarrow 3\lambda - 4\mu = k - 2$

Solving (A) + (B) simultaneously gives $\lambda = 0.6$, $\mu = 0.2$

the lines intersect so (C) must be consistent with these values of λ and μ

hence (C) $k = 3\lambda - 4\mu + 2 = 3(0.6) - 4(0.2) + 2$

$\Rightarrow k = 3$

G

(b) **In this question you must show detailed reasoning.**

Find the acute angle between the two lines. **[4]**

The angle between the lines is the angle between the direction vectors.

First we find the dot products of the vectors

$$\begin{pmatrix} -1 \\ 1 \\ 3 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix} = -1(2) + 1(3) + 3(4) = 13$$

Recall that $\cos\theta = \dfrac{\underline{a} \cdot \underline{b}}{|\underline{a}||\underline{b}|}$

So $\cos\theta = \dfrac{13}{\sqrt{1^2+1^2+3^2}\sqrt{2^2+3^2+4^2}}$

question asks for acute angle if $\theta > 90°$ we would need to give $180 - \theta$.

$\Rightarrow \theta = \arccos\left(\dfrac{13}{\sqrt{319}}\right) = 43.29\ldots = 43.3°$

A

9 A linear transformation of the plane is represented by the matrix $\mathbf{M} = \begin{pmatrix} 1 & -2 \\ \lambda & 3 \end{pmatrix}$, where λ is a constant.

(a) Find the set of values of λ for which the linear transformation has no invariant lines through the origin. **[5]**

We need to use $M\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x' \\ y' \end{pmatrix}$ to find the invariant lines.

$$\begin{pmatrix} 1 & -2 \\ \lambda & 3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x' \\ y' \end{pmatrix} \Rightarrow \begin{matrix} x - 2y = x' & (A) \\ \lambda x + 3y = y' & (B) \end{matrix}$$

So the point (x, y) on $y = mx$ is mapped to (x', y'). So $y' = mx'$ (lines are through origin so $c=0$)

(B) $\lambda x + 3y = y'$

using $y = mx$: $\lambda x + 3y = mx'$ using (A)

$\lambda x + 3(mx) = m(x - 2y)$

$\lambda x + 3mx = mx - 2m(mx)$

$\lambda x + 3mx = mx - 2m^2 x$

$2m^2 x + 2mx + \lambda x = 0$

$x(2m^2 + 2m + \lambda) = 0$

no invariant lines if $2m^2 + 2m + \lambda = 0$ has no real roots

$\Rightarrow$ det < 0

$2^2 - 4 \times 2 \times \lambda < 0$

$8\lambda > 4 \Rightarrow \lambda > \frac{1}{2}$

A

(b) Given that the transformation multiplies areas by 5 and reverses orientation, find the invariant lines. **[3]**

We can consider the determinant of M.
Since M multiplies areas by 5, det M has a magnitude of 5. It also has a negative sign as it reverses orientation.

So det M $= -5$

$3 - (-2)(\lambda) = -5 \Rightarrow 2\lambda = -8 \Rightarrow \lambda = -4$

Using part a), $2m^2 + 2m - 4 = 0$

$m^2 + m - 2 = 0$

$(m-1)(m+2) = 0$

$m = 1,\ m = -2$

So invariant lines are $y = x$ and $y = -2x$

R

10 In this question you must show detailed reasoning.

The region in the first quadrant bounded by curve $y = \cosh\frac{1}{2}x^2$, the y-axis, and the line $y = 2$ is rotated through 360° about the y-axis.

Find the exact volume of revolution generated, expressing your answer in a form involving a logarithm. **[7]**

Recall that the volume generated by rotating a curve about the y-axis 2π radians is given by

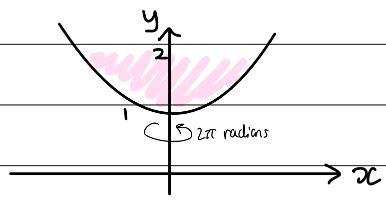
$$V = \pi\int_a^b x^2\, dy$$

Rearranging the equation for the curve $y = \cosh\left(\frac{1}{2}x^2\right)$

$$\frac{1}{2}x^2 = \operatorname{arcosh} y$$

$$x^2 = 2\operatorname{arcosh} y$$

By sketching $y = \cosh\left(\frac{1}{2}x^2\right)$, it is clear the bounds are $y = 1$ and $y = 2$



So volume $= \pi\int_0^2 2\operatorname{arcosh} y\, dy = 2\pi\int_0^2 \operatorname{arcosh} y\, dy$

We need to do this integral by parts

let $u = \operatorname{arcosh} y$ $\quad \frac{du}{dx} = \frac{1}{\sqrt{y^2-1}}$

$\frac{dv}{dy} = 1$ $\quad v = y$

so $\int \operatorname{arcosh} y\, dy = y\operatorname{arcosh} y - \int \frac{y}{\sqrt{y^2-1}}\, dy$

by recognition (could also be done by substitution)

$= y\operatorname{arcosh} y - \sqrt{y^2-1} + c$

hence volume $= 2\pi\left[y\operatorname{arcosh} y - \sqrt{y^2-1}\right]_1^2$

$= 2\pi\left(2\operatorname{arcosh} 2 - \sqrt{2^2-1} - \left(\operatorname{arcosh} 1 - \sqrt{1^2-1}\right)\right)$

$= 2\pi\left(2\ln\left(2 + \sqrt{2^2-1}\right) - \sqrt{3} - 0 + 0\right)$

$= 2\pi\left(2\ln\left(2+\sqrt{3}\right) - \sqrt{3}\right)$

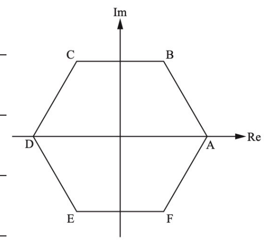
$= 4\pi\ln\left(2+\sqrt{3}\right) - 2\pi\sqrt{3}$ units^3

11 In this question you must show detailed reasoning.

In Fig. 11, the points A, B, C, D, E and F represent the complex sixth roots of 64 on an Argand diagram. The midpoints of AB, BC, CD, DE, EF and FA are G, H, I, J, K and L respectively.

A

(a) Write down, in exponential ($re^{i\theta}$) form, the complex numbers represented by the points A, B, C, D, E and F. **[2]**



To find r, $r^6 = 64 \Rightarrow r = \sqrt[6]{64} = 2$

Since Fig. 11 is a regular hexagon, every interior angle is $\frac{\pi}{3}$ rad.

Hence $A = 2e^{0i} = 2$ $\quad E = 2e^{\frac{4\pi}{3}i}$

$B = 2e^{\frac{\pi}{3}i}$ $\quad F = 2e^{\frac{5\pi}{3}i}$

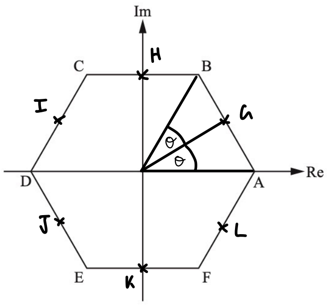
$C = 2e^{\frac{2\pi}{3}i}$

$D = 2e^{\pi i} = -2$

A

(b) When these complex numbers are multiplied by the complex number w, the resulting complex numbers are represented by the points G, H, I, J, K and L.

Find w in exponential form. **[4]**

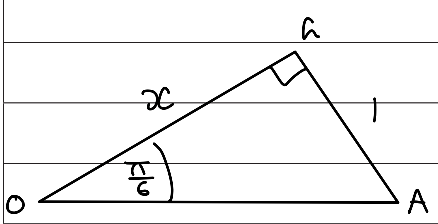


Multiplying a complex number by $w = re^{i\theta}$ rotates the number by θ and scales it's magnitude by r.

From the diagram, $\angle AOB = \frac{\pi}{3}$ rad

so $\angle AOG = \frac{\pi}{6}$ rad

hence $\theta = \frac{\pi}{6}$



$\tan\frac{\pi}{6} = \frac{1}{x} \rightarrow x = \frac{1}{1/\sqrt{3}} = \sqrt{3}$

'Scaling factor' 'new' modulus

hence $2 \times r = \sqrt{3} \Rightarrow r = \frac{\sqrt{3}}{2}$

'old' modulus so $w = \frac{\sqrt{3}}{2}e^{\frac{\pi}{6}i}$

A

(c) You are given that G, H, I, J, K and L represent roots of the equation $z^6 = p$.

Find p. **[2]**

From above, $G = z = \sqrt{3}e^{\frac{\pi}{6}i} = \sqrt{3}\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)$

So $p = \left(\sqrt{3}\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)\right)^6$ by de Moivre's if $z = r(\cos\theta + i\sin\theta)$, $z^n = r^n(\cos n\theta + i\sin n\theta)$

$= 27(\cos\pi + i\sin\pi)$

$= 27(-1 + i(0))$

$= -27$

$\Rightarrow p = -27$

G

12 (a) Given that $z = \cos\theta + \mathrm{i}\sin\theta$, express $z^n + \frac{1}{z^n}$ and $z^n - \frac{1}{z^n}$ in simplified trigonometric form. **[2]**

Recall that $\cos n\theta = \dfrac{z^n + z^{-n}}{2}$

$$\Rightarrow z^n + \frac{1}{z^n} = 2\cos n\theta$$

$$\sin n\theta = \frac{z^n - z^{-n}}{2i}$$

$$\Rightarrow z^n - \frac{1}{z^n} = 2i\sin n\theta$$

A

(b) By considering $\left(z+\frac{1}{z}\right)^3\left(z-\frac{1}{z}\right)^3$, find constants A and B such that

$\sin^3\theta\cos^3\theta = A\sin 6\theta + B\sin 2\theta$. **[6]**

Firstly let's use part a) to rewrite $\left(z+\frac{1}{z}\right)^3\left(z-\frac{1}{z}\right)^3$

$$\left(z+\frac{1}{z}\right)^3\left(z-\frac{1}{z}\right)^3 = (z+z^{-1})^3(z-z^{-1})^3$$

using a)

$$= (2\cos\theta)^3(2i\sin\theta)^3$$
$$= 8\cos^3\theta \times -8i\sin^3\theta$$
$$= -64i\cos^3\theta\sin^3\theta \quad (A)$$

Now let's expand out $\left(z+\frac{1}{z}\right)^3\left(z-\frac{1}{z}\right)^3$ using a binomial expansion.

$$\left(z+\frac{1}{z}\right)^3\left(z-\frac{1}{z}\right)^3 = (z+z^{-1})^3(z-z^{-1})^3$$
$$= (z^3 + 3z + 3z^{-1} + z^{-3})(z^3 - 3z + 3z^{-1} - z^{-3})$$

	z^3	$-3z$	$+3z^{-1}$	$-z^{-3}$
z^3	z^6	$\cancel{-3z^4}$	$+3z^2$	$\cancel{-1}$
$3z$	$\cancel{+3z^4}$	$-9z^2$	$\cancel{+9}$	$-3z^{-2}$
$3z^{-1}$	$3z^2$	$\cancel{-9}$	$+9z^{-2}$	$\cancel{-3z^{-4}}$
z^{-3}	$\cancel{1}$	$-3z^{-2}$	$\cancel{+3z^{-4}}$	$-z^{-6}$

$$= z^6 - 3z^2 + 3z^{-2} - z^{-6}$$
$$= z^6 - z^{-6} - 3z^2 + 3z^{-2}$$
$$= z^6 - z^{-6} - 3(z^2 - z^{-2})$$

using a)

$$= 2i\sin 6\theta - 3(2i\sin 2\theta)$$
$$= 2i\sin 6\theta - 6i\sin 2\theta \quad (B)$$

So using (A) and (B),

$$-64i\cos^3\theta\sin^3\theta = 2i\sin 6\theta - 6i\sin 2\theta$$
$$\cos^3\theta\sin^3\theta = -\frac{1}{32}\sin 6\theta + \frac{3}{32}\sin 2\theta$$

$\left(A = -\frac{1}{32} \text{ and } B = \frac{3}{32}\right)$

G

13 (a) Using exponentials, prove that $\sinh 2x = 2\cosh x \sinh x$. **[2]**

Considering the LHS, $\sinh 2x = \dfrac{e^{2x} - e^{-2x}}{2}$

Recall $\cosh x = \dfrac{e^{x} + e^{-x}}{2}$ and $\sinh x = \dfrac{e^{x} - e^{-x}}{2}$

Considering the RHS,

$2\cosh x \sinh x = 2\left(\frac{1}{2}\right)(e^{x} + e^{-x})\left(\frac{1}{2}\right)(e^{x} - e^{-x})$

$= \frac{1}{2}(e^{x} + e^{-x})(e^{x} - e^{-x}) = \frac{1}{2}(e^{2x} - 1 + 1 - e^{-2x})$

$= \dfrac{e^{2x} - e^{-2x}}{2} = \sinh 2x$ (= LHS as shown above)

A

(b) Hence show that if $f(x) = \sinh^2 x$, then $f''(x) = 2\cosh 2x$. **[2]**

$f(x) = (\sinh x)^2$ $\quad \frac{d}{dx}(\sinh x) = \cosh x$

$= 2 \times \sinh x \times \cosh x = 2\sinh x \cosh x = \sinh 2x$

chain rule

$f''(x) = 2\cosh 2x$ // as required

E

(c) Explain why the coefficients of odd powers in the Maclaurin series for $\sinh^2 x$ are all zero. **[2]**

Since $f''(x) = 2\cosh 2x$, subsequent odd derivatives are integer multiples of $\sinh 2x$ (since $\frac{d}{dx}(\cosh x) = \sinh x$)

Further, if $x = 0$, $\sinh x = \dfrac{e^0 - e^{-0}}{2} = \dfrac{1-1}{2} = 0$

So $f^n(x) = 0$ if n is odd.

So as any odd powers in the series are multiplied by 0, all the odd powers in the series are zero.

R

(d) Find the coefficient of x^n in this series when n is a positive even number. **[3]**

First find a general form for $f^n(0)$ if n is even.

$f^2(x) = 2\cosh 2x \Rightarrow f^2(0) = 2\cosh 0 = 2$

$f^3(x) = 4\sinh 2x$

$f^4(x) = 8\cosh 2x \Rightarrow f^4(0) = 8\cosh 0 = 8$

$\Rightarrow f^n(0) = 2^{n-1}$

From the FB, the general term in a maclaurin series is $\dfrac{f^{(r)}(0)}{r!}x^r$. So if n is even, term is $\dfrac{f^n(0)}{n!}x^n = \dfrac{2^{n-1}}{n!}x^n$ //

R

14 Solve the simultaneous differential equations

A $\frac{dx}{dt} + 2x = 4y$, B $\frac{dy}{dt} + 3x = 5y$,

given that when $t = 0$, $x = 0$ and $y = 1$. **[11]**

This is a set of coupled differential equations.
To solve this, we first need to eliminate y to find a second order differential equation in x.

A $\frac{dx}{dt} + 2x = 4y$

$\frac{d}{dt}\left(\frac{dx}{dt}\right) + \frac{d}{dt}(2x) = \frac{d}{dt}(4y)$ differentiating implicitly with respect to t.

$\frac{d^2x}{dt^2} + 2\frac{dx}{dt} = 4\frac{dy}{dt}$ now we eliminate $\frac{dx}{dt}$ and $\frac{dy}{dt}$

Using A, $\frac{dx}{dt} = 4y - 2x$ Using B, $\frac{dy}{dt} = 5y - 3x$

Substituting these in,

$$\frac{d^2x}{dt^2} + 2(4y - 2x) = 4(5y - 3x)$$

$$\frac{d^2x}{dt^2} + 8y - 4x = 20y - 12x$$

$$\frac{d^2x}{dt^2} - 12y + 8x = 0$$ $4y = \frac{dx}{dt} + 2x$

$$\frac{d^2x}{dt^2} - 3\left(\frac{dx}{dt} + 2x\right) + 8x = 0$$

$$\frac{d^2x}{dt^2} - 3\frac{dx}{dt} + 2x = 0$$

(we keep substituting A and B until our DE is in a 'solvable' 2nd order form)

(answer space continued on next page)

14 **(continued)**

$$\frac{d^2x}{dt^2} - 3\frac{dx}{dt} + 2x = 0$$

Auxilliary Equation: $\lambda^2 - 3\lambda + 2 = 0$

$$(\lambda - 2)(\lambda - 1) = 0$$

$$\lambda = 2, \ \lambda = 1$$

Since we have 2 real roots, our general form is

$$x = Ae^t + Be^{2t}$$

now we need to also find $y = f(t)$

using Ⓐ, $y = \frac{1}{4}\left(\frac{dx}{dt} + 2x\right)$

$$\frac{dx}{dt} = Ae^t + 2Be^{2t}$$

$$\text{so} \quad y = \frac{1}{4}\left(Ae^t + 2Be^{2t} + 2\left(Ae^t + Be^{2t}\right)\right)$$

$$y = \frac{1}{4}\left(Ae^t + 2Ae^{2t} + 2Be^t + 2Be^{2t}\right)$$

$$y = \frac{3}{4}Ae^t + Be^{2t}$$

so $x = Ae^t + Be^{2t}$ and $y = \frac{3}{4}Ae^t + Be^{2t}$

Finally we need to find A and B using the given values.

When $t = 0$, $x = 0$: $0 = Ae^0 + Be^0 \Rightarrow A + B = 0$ ①

When $t = 0$, $y = 1$: $1 = \frac{3}{4}Ae^0 + Be^0 \Rightarrow \frac{3}{4}A + B = 1$

$$3A + 4B = 4 \quad ②$$

Solving ① and ② simultaneously gives

$$A = -4 \text{ and } B = 4$$

hence $x = -4e^t + 4e^{2t}$

and $y = -3e^t + 4e^{2t}$

15 **(a)** Show that the three planes with equations

$$x + \lambda y + 3z = -12$$
$$2x + y + 5z = -11$$
$$x - 2y + 2z = -9$$

A

where λ is a constant, meet at a unique point except for one value of λ which is to be determined. **[3]**

We can rewrite this as

$$\underbrace{\begin{pmatrix} 1 & \lambda & 3 \\ 2 & 1 & 5 \\ 1 & -2 & 2 \end{pmatrix}}_{\hookrightarrow M} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -12 \\ -11 \\ -9 \end{pmatrix}$$

First let's find det M.

$$\det M = 1\begin{vmatrix} 1 & 5 \\ -2 & 2 \end{vmatrix} - \lambda\begin{vmatrix} 2 & 5 \\ 1 & 2 \end{vmatrix} + 3\begin{vmatrix} 2 & 1 \\ 1 & -2 \end{vmatrix}$$

$$= 1(2+10) - \lambda(4-5) + 3(-4-1) = \lambda - 3$$

The planes do not meet at a unique if $\det M = 0$ as M is singular. So $\lambda - 3 = 0 \Rightarrow \lambda = 3$

So if $\lambda = 3$ the planes do not meet at a unique point.

A

(b) In the case $\lambda = -2$, use matrices to find the point of intersection P of the planes, showing your method clearly. **[3]**

Using a calculator,

$$M^{-1} = \begin{pmatrix} -12/5 & 2/5 & 13/5 \\ -1/5 & 1/5 & -1/5 \\ 1 & 0 & -1 \end{pmatrix}$$

So $\begin{pmatrix} 1 & -2 & 3 \\ 2 & 1 & 5 \\ 1 & -2 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -12 \\ -11 \\ -9 \end{pmatrix}$

$(\times M^{-1})$ $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -12/5 & 2/5 & 13/5 \\ -1/5 & 1/5 & -1/5 \\ 1 & 0 & -1 \end{pmatrix} \begin{pmatrix} -12 \\ -11 \\ -9 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ -3 \end{pmatrix}$

hence $p = (1, 2, -3)$

G

The line l has equation $\dfrac{x-1}{2} = \dfrac{y-1}{-1} = \dfrac{z+2}{-2}$.

(c) Find a vector equation of l. **[2]**

Each of the terms in the equation for l is equal to a scalar parameter, say λ.

$\dfrac{x-1}{2} = \lambda \Rightarrow x = 1 + 2\lambda$

$\dfrac{y-1}{-1} = \lambda \Rightarrow y = 1 - \lambda \Rightarrow \underline{r} = \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ -1 \\ -2 \end{pmatrix}$

$\dfrac{z+2}{-2} = \lambda \Rightarrow z = -2 - 2\lambda$

A

(d) Find the shortest distance between the point P and l. **[4]**

Recall that the shortest distance between a line ($\underline{r} = \underline{a} + \lambda \underline{d}$) and a point P is given by $\dfrac{|\vec{aP} \times \underline{d}|}{|\underline{d}|}$

$\vec{aP} = \underline{p} - \underline{a} = \begin{pmatrix} 1 \\ 2 \\ -3 \end{pmatrix} - \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$

$\vec{aP} \times \underline{d} = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} \times \begin{pmatrix} 2 \\ -1 \\ -2 \end{pmatrix} = \begin{pmatrix} -3 \\ -2 \\ -2 \end{pmatrix}$

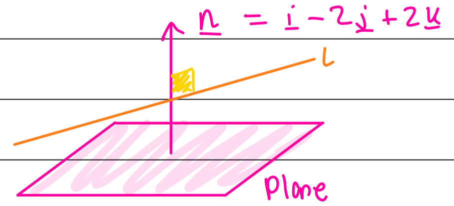
$|\vec{aP} \times \underline{d}| = \sqrt{3^2 + 2^2 + 2^2} = \sqrt{17}$

$|\underline{d}| = \sqrt{2^2 + 1^2 + 2^2} = 3$

hence distance $= \dfrac{\sqrt{17}}{3}$ so $\dfrac{\sqrt{17}}{3}$ units

A

(e) (i) Show that l is parallel to the plane $x-2y+2z=-9$. **[3]**



As seen to the left, if L is parallel to the plane, L is perpendicular to the normal to the plane.

L is perpendicular to the normal if the scalar product of the direction vector of L and the normal to the plane is 0.

$$\begin{pmatrix}1\\-2\\2\end{pmatrix}\cdot\begin{pmatrix}2\\-1\\-2\end{pmatrix} = 1(2)-2(-1)+2(-2) = 0$$

$\therefore$ L is parallel to plane

A

(ii) Find the distance between l and the plane $x-2y+2z=-9$. **[2]**

Since the plane and line are parallel, the distance between them is always the same.

So we can just find the distance between P (which lies on the plane) and L.

In the formula booklet, the distance between (x_1, y_1, z_1) and $n_1x + n_2y + n_3z + d = 0$ is

$$\frac{|n_1x_1 + n_2y_1 + n_3z_1 + d|}{\sqrt{n_1^2 + n_2^2 + n_3^2}}$$

$$\text{distance} = \frac{|1(1) + 1(-2) + 2(-2) + 9|}{\sqrt{1^2 + 2^2 + 2^2}} = \frac{4}{3}$$

hence distance between L and plane is $\frac{4}{3}$ units

G

16 The population density P, in suitable units, of a certain bacterium at time t hours is to be modelled by a differential equation. Initially, the population density is zero, and its long-term value is A.

(a) One simple model is to assume that the rate of change of population density is directly proportional to $A - P$.

(i) Formulate a differential equation for this model. **[1]**

The rate of change of population density is represented by $\frac{dP}{dt}$.

Hence $\frac{dP}{dt} \propto (A - P) \Rightarrow \frac{dP}{dt} = k(A - P)$

A

16(a)(ii) **(ii)** Verify that $P = A(1 - e^{-kt})$, where k is a positive constant, satisfies

- this differential equation,
- the initial condition,
- the long-term condition. **[3]**

1. The Differential Equation

$\frac{dP}{dt} = A(ke^{-kt})$

So $\frac{dP}{dt} = Ak\left(1 - \frac{P}{A}\right)$

$\frac{dP}{dt} = k(A - P)$ $\therefore$ satisfied

$P = A(1 - e^{-kt})$

$P = A - Ae^{-kt}$

$Ae^{-kt} = A - P$

$e^{-kt} = 1 - \frac{P}{A}$

2. The intial condition

When $t = 0$, $P = A(1 - e^0) = 0$ $\therefore$ satisfied

3. The long-term condition

as $t \to \infty$, $e^{-kt} \to 0$. so $P \to A(1 - 0)$

$P \to A$ $\therefore$ satisfied

An alternative model uses the differential equation

$$\frac{dP}{dt} - \frac{P}{t(1+t^2)} = Q(t),$$

where $Q(t)$ is a function of t.

R

(b) Find the integrating factor for this differential equation, showing that it can be written in the form $\frac{\sqrt{1+t^2}}{t}$. **[8]**

So the differential equation, $\frac{dP}{dt} - \frac{P}{t(1+t^2)} = Q(t)$, is in standard linear 1st order DE form.

The integrating factor, $I(t)$, is given by

$$I(t) = e^{\int -\frac{1}{t(1+t^2)}\,dt}$$

$\int \frac{1}{t(1+t^2)}\,dt$ can be found using partial fractions.

$$\frac{1}{t(1+t^2)} = \frac{A}{t} + \frac{Bt+C}{t^2+1} = \frac{A(t^2+1) + t(Bt+C)}{t(t^2+1)}$$

So $At + At^2 + A + Bt^2 + Ct = 1$

$A = 1$, $A + B = 0$, $A + C = 0$

$B = -1$, $C = 0$

hence $\int \frac{1}{1(1+t^2)}\,dt = \int \frac{1}{t} - \frac{t}{t^2+1}\,dt$ using $\int \frac{f'(x)}{f(x)}\,dx = \ln|f'(x)|$

$= \ln|t| - \frac{1}{2}\ln|t^2+1|$ (no $+c$ required, as multiplying the whole DE by $+c$ will cancel it out)

$= \ln|t| - \ln|\sqrt{1+t^2}|$

$= \ln\left|\frac{t}{\sqrt{1+t^2}}\right|$

So $I(x) = e^{-\ln\left|\frac{t}{\sqrt{1+t^2}}\right|} = e^{\ln\left|\frac{\sqrt{1+t^2}}{t}\right|} = \frac{t}{\sqrt{1+t^2}}$ as required.

A

(c) Suppose that $Q(t) = 0$.

(i) Show that $P = \frac{At}{\sqrt{1+t^2}}$. **[4]**

The DE is now $\frac{dP}{dt} - \frac{P}{t(1+t^2)} = 0$

This is solved by first multiplying through by $I(t)$.

$\times I(t)$: $\frac{\sqrt{1+t^2}}{t}\frac{dP}{dt} - \frac{\sqrt{1+t^2}}{t} \times \frac{P}{t(1+t^2)} = 0$

$$\frac{\sqrt{1+t^2}}{t}\frac{dP}{dt} - \frac{P}{t^2\sqrt{1+t^2}} = 0$$

The LHS is implicit product rule. $u = P$ $u' = \frac{dP}{dt}$

$v = \frac{\sqrt{1+t^2}}{t}$ $v' = -\frac{1}{t^2\sqrt{1+t^2}}$

$$\frac{d}{dt}\left(\frac{Pt}{\sqrt{1+t^2}}\right) = 0$$

$$\frac{Pt}{\sqrt{1+t^2}} = \int 0 \, dt \implies \frac{Pt}{\sqrt{1+t^2}} = K$$

as $t \to \infty$, $P \to K$. hence $K = A$.

so $\frac{Pt}{\sqrt{1+t^2}} = A \implies P = \frac{At}{\sqrt{1+t^2}}$

G

(ii) Find the time predicted by this model for the population density to reach half its long-term value. Give your answer correct to the nearest minute. **[2]**

Half of the long term value of P is $\frac{1}{2}A$.

So we need to solve $P = \frac{1}{2}A$

$$\frac{1}{2}\cancel{A} = \frac{\cancel{A}t}{\sqrt{1+t^2}}$$

$$\sqrt{1+t^2} = 2t$$

$$1 + t^2 = 4t^2$$

$$3t^2 = 1$$

$$t^2 = \frac{1}{3}$$

$$t = \pm\frac{1}{\sqrt{3}}$$

so $0.577\ldots$ hours $= 34.64\ldots = 35$ minutes

R

(d) Now suppose that $Q(t) = \dfrac{te^{-t}}{\sqrt{1+t^2}}$.

Show that $P = \dfrac{At - te^{-t}}{\sqrt{1+t^2}}$. [You may assume that $\lim_{t\to\infty} te^{-t} = 0$.] **[5]**

The DE is now $\dfrac{dP}{dt} - \dfrac{P}{t(1+t^2)} = \dfrac{te^{-t}}{\sqrt{1+t^2}}$

This is solved by first multiplying through by $I(t)$.

$\times I(t)$: $\dfrac{\sqrt{1+t^2}}{t}\dfrac{dP}{dt} - \dfrac{\sqrt{1+t^2}}{t} \times \dfrac{P}{t(1+t^2)} = \dfrac{\cancel{\sqrt{1+t^2}}}{\cancel{t}} \times \dfrac{\cancel{t}e^{-t}}{\cancel{\sqrt{1+t^2}}}$

$$\frac{\sqrt{1+t^2}}{t}\frac{dP}{dt} - \frac{P}{t^2\sqrt{1+t^2}} = e^{-t}$$

The LHS is implicit product rule. $u = P$ $\quad u' = \dfrac{dP}{dt}$

$v = \dfrac{\sqrt{1+t^2}}{t}$ $\quad v' = -\dfrac{1}{t^2\sqrt{1+t^2}}$

so $\dfrac{d}{dt}\left(\dfrac{Pt}{\sqrt{1+t^2}}\right) = e^{-t}$

$$\frac{Pt}{\sqrt{1+t^2}} = \int e^{-t}\,dt \Rightarrow \frac{Pt}{\sqrt{1+t^2}} = -e^{-t} + C$$

$$\Rightarrow P = \frac{Ct - te^{-t}}{\sqrt{1+t^2}}$$

as $t \to \infty$, $P \to C$ as $te^{-t} \to 0$. hence $C = A$

so $P = \dfrac{At - te^{-t}}{\sqrt{1+t^2}}$ as required

A

It is found that the long-term value of P is 10, and P reaches half this value after 37 minutes.

(e) Determine which of the models proposed in parts **(c)** and **(d)** is more consistent with these data. **[2]**

Here we need to investigate each model. We need to find P when $t = \frac{37}{60}$ for each.

Model One: $P = \dfrac{10\left(\frac{37}{60}\right)}{\sqrt{1+\left(\frac{37}{60}\right)^2}} = 5.248\ldots$

Model Two: $P = \dfrac{10\left(\frac{37}{60}\right) - \left(\frac{37}{60}\right)e^{-\left(\frac{37}{60}\right)}}{\sqrt{1+\left(\frac{37}{60}\right)^2}} = 4.96\ldots$

Hence the 2nd model is better as P is closer to $\frac{10}{2} = 5$.

ADDITIONAL ANSWER SPACE

If additional space is required, you should use the following lined page(s). The question number(s) must be clearly shown in the margin(s).